# Tutorial Math 2040 A 2nd Week

3.2 Rank and inverse

5. compute rank and inverse if exists

c) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 2 & 3 & -1 \end{pmatrix}$ rank 2 $\Rightarrow$ no inverse

d) $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}$ rank 3 inverse $= \begin{pmatrix} -\frac{1}{2} & 3 & -1 \\ \frac{3}{2} & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix}$

e) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ rank 3 inverse $= \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{pmatrix}$

( details are shown in class )

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right) \rightarrow \cdots$$

3.3 Linear Equations

2. find the dim of and a basis for the solution set

a) $x_1 + 3x_2 = 0$ $\quad\{(-3, 1)\}$ is a basis $\quad$ dim $= 1$

$2x_1 + 6x_2 = 0$

b) $x_1 + 2x_2 - x_3 = 0$ $\quad\{(-1, 1, 1)\}$ is a basis $\quad$ dim $= 1$

$2x_1 + x_2 + x_3 = 0$

e) $x_1 + 2x_2 - 3x_3 + x_4 = 0$ $\quad\{(-2, 1, 0, 0)\ (3, 0, 1, 0), (-1, 0, 0, 1)\}$

is a basis $\quad$ dim $= 3$.

f) $x_1 + 2x_2 = 0$ $\quad\{(0, 0)\}$ is a basis $\quad$ dim $= 0$

$x_1 - x_2 = 0$

( details are shown in class)

4. Solve systems of linear Equations

a) $x_1 + 3x_2 = 4$

$2x_1 + 5x_2 = 3$

$A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} -5 & 3 \\ 2 & -1 \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -11 \\ 5 \end{pmatrix}$

b) $x_1 + 2x_2 - x_3 = 5$

$x_1 + x_2 + x_3 = 1$

$2x_1 - 2x_2 + x_3 = 4$

$A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \\ 2 & -2 & 1 \end{pmatrix}$ $A^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{9} & \frac{1}{3} & -\frac{2}{9} \\ -\frac{4}{9} & \frac{2}{3} & -\frac{1}{9} \end{pmatrix}$ $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$

3.4 Computations

12. Let $V$ be the set of all solutions to

$x_1 - x_2 + 2x_4 - 3x_5 + x_6 = 0$

$2x_1 - x_2 - x_3 + 3x_4 - 4x_5 + 4x_6 = 0$

a) Show that $S = \{(0,-1,0,1,1,0), (1,0,1,1,1,0)\}$ is a lin. ind. subset of $V$

b) Extend $S$ to a basis for $V$

Solution: a) Two vectors are indeed in $V$.

$a(0,-1,0,1,1,0)+b(1,0,1,1,1,0)=0 \Rightarrow a=b=0$

i.e. two vectors are lin. ind.

b) Find a basis, say, $\{(1,1,1,0,0,0), (-1,1,0,1,0,0)$

$(1,-2,0,0,1,0), (-3,-2,0,0,0,1)\}$

$$\begin{pmatrix} 0 & 1 & 1 & -1 & 1 & -3 \\ -1 & 0 & 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \{(0,-1,0,1,1,0), (1,0,1,1,1,0), (-1,1,0,1,0,0),$

$(-3,-2,0,0,0,1)\}$ form a basis for $V$.

Remark: In b), we first find ANY basis for $V$, say, $\{v_1 \cdots v_4\}$.

Then we add $w_1$ and $w_2$ into this set, to get $\{w_1, w_2, v_1 \cdots v_4\}$.

Finally, we find the maximal subset of lin. ind vectors in

$\{w_1, w_2, v_1 \cdots v_4\}$ that contains $w_1$ and $w_2$.

# Sec 2.5 The Change of Coordinate Matrix

6. $A$ – matrix $\beta$ – ordered basis find $[L_A]_\beta$ and an invertible matrix $Q$ s.t. $[L_A]_\beta = Q^{-1}AQ$.

a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

$Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $[L_A]_\beta = \begin{pmatrix} 6 & 11 \\ -2 & -4 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$

$Q = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $[L_A]_\beta = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$

10. If $A$ and $B$ are $n \times n$ matrix that are similar to each other, then $\operatorname{tr} A = \operatorname{tr} B$

Solution: Prove "by hand" that $\operatorname{tr} AB = \operatorname{tr} BA$

$$\operatorname{tr} B = \operatorname{tr}(Q^{-1}AQ) = \operatorname{tr}(QQ^{-1}A) = \operatorname{tr} A$$

Remark: $\operatorname{tr} ABC = \operatorname{tr} CAB = \operatorname{tr} BCA$

But, in general, we do not know $\operatorname{tr} ABC = \operatorname{tr} ACB$.
(Most of the time, this is wrong)